SHORT COMMUNICATION

A psychophysical theory of Shannon entropy

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Abstract

Connections between information theory and decision under uncertainty have been attracting attention in econophysics, neuroeconomics and quantum decision theory. This paper proposes a psychophysical theory of Shannon entropy based on a mathematical equivalence of delay and uncertainty in decision-making, and psychophysics of the perception of waiting time in probabilistic choices. Furthermore, it is shown that the well-known Shannon entropy is a special case of the general psychophysical entropy. Future directions in the application of the present theory to studies in econophysics and neuroeconomics are discussed.

1. INTRODUCTION

Decision under risk and probabilistic uncertainty (probabilistic choice) has been a major topic in microeconomics (e.g., the expected utility theory, von Neumann & Morgenstern 1947), behavioral neuroeconomics (e.g. the prospect theory, Kahneman & Tversky 1977), and econophysics (Anteneodo *et al.* 2002). Studies in behavioral and neuro-economics have revealed that humans and non-human animals discount the value of probabilistic rewards as the receipt becomes more uncertain ("probability discounting", Rachlin & Raineri 1991). In order to develop decision theory on probabilistic choice, recent efforts in econophysics have started to combine probabilistic choice processes with intertemporal choice process (Takahashi 2007). Delay discounting in intertemporal choice refers to the devaluation of a delayed reward compared to the value of a sooner reward. In this line of investigations into the unification of delay and probability discounting, recent studies in behavioral psychology and neuroeconomics have demonstrated a mathematical equivalence of delay and probabilistic uncertainty (risk) in reward-seeking behavior (Takahashi 2007). Furthermore, Takahashi (2005, 2006) proposed that perception of waiting time in intertemporal choice is logarithmic in physical time, which has recently been confirmed experimentally (Takahashi *et al.* 2008; Zauberman *et al.* 2009).

I introduce, in this paper, a novel theory for Shannon information entropy by utilizing psychophysics of waiting time based on the equivalence of delay and uncertainty (Rachlin & Raineri 1991; Takahashi 2007). Notably, I derive generalized Shannon entropy based on the psychophysical theory and demonstrate that conventional Shannon entropy is a special case for it.

This paper is organized in the following manner. In Section 2, I briefly introduce the mathematical equivalence of delay and probabilistic uncertainty, and psychophysics of logarithmic waiting time perception. In Section 3, I explain the psychophysical theory of information entropy. In Section 4, some conclusions from this study and future study directions by utilizing the present psychophysical theory of information entropy in neuroeconomics and neuropsychiatry of risky behavior are discussed.

2. A MATHEMATICAL EQUIVALENCE OF DELAY AND UNCERTAINTY IN DECISION MAKING

As noted above, it has been demonstrated that delay until receipt of gains in intertemporal choice and uncertainty of winning of gains in probabilistic choice may be equivalent. In unifying decision over time and under uncertainty, Rachlin *et al.* (1991) hypothesized that a decrease in a probability of winning an uncertain reward corresponds to an increase in a delay until winning the reward. Specifically, an average waiting time until winning an uncertain reward is proportional to $(1/p)-1$ ("odds against"), where p is a probability of winning the uncertain reward. Therefore, according to Rachlin and colleagues' hypothesis, decision-making models in intertemporal choice (delay discounting) can straightforwardly be extended into probabilistic choice(s), after replacing a parameter of delay in intertemporal choice models with the odds against parameter. Hence, it appears to be a promising direction to establish information theory on the psychophysical equivalence of delay and probability (in terms of the odds-against = $[(1-p)/p]$ in decision under probabilistic uncertainty (risk).

With respect to psychophysics of time-perception, Takahashi (2005, 2006) proposed that "psychological time during intertemporal choice" ($=\tau$) follows the Weber-Fechner law:

$$\tau(\mathrm{D}) = \mathrm{a} \ln (1+\mathrm{b}\,\mathrm{D}), \tag{1}$$

where $\tau \in [0,\infty]$ is perceived/anticipated psychological time of physical delay $\mathrm{D} \in [0,\infty]$, and a and $\mathrm{b} \in [0,\infty]$ are free parameters (note that b is a nondimensionalization coefficient of physical time D). The theoretical proposal has been confirmed experimentally (Takahashi *et al.* 2008; Zauberman *et al.* 2009). Together, it may be a promising direction how information theory in decision theory and neuroeconomics could be formalized in terms of psychophysics of waiting time in probabilistic choices when D is proportional to the odds against $[(1-p)/p]$.

3. PSYCHOPHYSICS OF INFORMATION ENTROPY

Let us suppose that the (average) inter-trial time in probabilistic choices is t. Then, the (average) waiting time in probabilistic choices for a single outcome is

$$\mathrm{D} = \left(\frac{1-p}{p}\right)\mathrm{t}. \tag{2}$$

Therefore, psychological waiting time is (from Equation 1),

$$\tau(\mathrm{D}) = \mathrm{a} \ln \left(1+\mathrm{bt}\left(\frac{1-p}{p}\right)\right). \tag{3}$$

Let us now consider the case with multiple N outcomes (x_i, p_i) where x_i, p_i are the magnitude of the i^{th} outcome and the probability of obtaining the outcome x_i (i is an integer $\in [1, N]$). Let us denote

$$\tau_i(D) = a_i \ln \left(1+b_i t_i \left(\frac{1-p_i}{p_i}\right)\right), \tag{4}$$

where the suffix i indicates each outcome i.

In this case, the average of anticipatory waiting time for N outcomes is

$$\mathrm{H_p} = \sum_{i=1}^{N} p_i \tau_i(D). \tag{5}$$

By inserting Equation 4 into 5, we obtain

$$\mathrm{H_p} = p_i \sum_{i=1}^{N} a_i \ln \left[1+b_i t_i \left(\frac{1-p_i}{p_i}\right)\right]. \tag{6}$$

In the special case for Equation 6 in which $a_i = b_i t_i = 1$, we obtain

$$\mathrm{H_s} = -p_i \sum_{i=1}^{N} \ln p_i ,$$

which is (conventional) Shannon information entropy (Shannon 1948). Therefore, it is demonstrated that the conventional Shannon entropy is a special case for the generalized psychophysical entropy (Equation 6).

4. CONCLUSIONS AND IMPLICATIONS FOR NEUROECONOMICS AND ECONOPHYSICS

Decision under probabilistic uncertainty (risk) has been attracting strong attention in neuroeconomics (Paulus *et al.* 2006; Weber & Huettel 2008; Hsu *et al.* 2009; Christopoulos *et al.* 2009; Gianotti *et al.* 2009; Levy *et al.* 2010). However, neuroeconomic studies on risk have not paid enough attention to information theory, while mainstream neuroscience has been strongly connected to information theory (e.g., Nemenman 2008). A recent econophysical study introduced a model based on Tsallis' entropy (Takahashi 2009). Therefore, future studies in econophysics and neuroeconomics should more extensively utilize information theory by adopting the present psychophysical theory of information entropy, because psychophysical theory has also been widely exploited in theoretical neuroscience (Dayan & Abbott 2001). These lines of studies may help understand neuroeconomic and neurochemical bases of risky decision making by neuropsychiatric patients (Paulus 2007) such as bipolar disorder patients (Chandler *et al.* 2009).

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